ON THE METHOD OF INVARIANT TRANSFORMATIONS OF THE GAS-DYNAMICS EQUATIONS

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Ovsiannikov [1] established the transformation of the desired functions and independent variables in the differential equations of the motion of an inviscid and non-heat-conducting gas, for which these equations retain their form. This transformation is considered in detail in [2 and 3]. If some exact solution of the gas dynamics equations is known, the transformation formulas afford the oportunity of obtaining a new exact solution of the same equations, and of thereby comparing some other E described by the new solution to the given gas flow E'. The flow E should have either the same or a higher number of dimensions as compared with the flow E', whereupon the method turns out to be applicable only for specific values of the adiabatic index, depending on the number of dimensions of the flow E and equal to $5/3$ for three-dimensional, 2 for two-dimensional and 3 for one-dimensional flow.

The density and pressure in the flow E turn out to be inversely proportional to some power of $t-c$, where t is the time, and c an arbitrary constant with the dimensionality of time; hence, the results of [2 and 3 for conventional gas dynamics are extended to the dynamics of an expanding gas.

If we hence obtain a flow E with a larger number of dimensions from a given one- or two-dimensional flow E', the additional velocity components turn out to be equal to the quotient of a division of the appropriate coordinates by the difference $t-c$. Proceeding from here, we can generalize the results of [2 and 3 somewhat by establishing the existence of such a flow E' as may be compared to the solution yielding some flow E in a range of a lesser number of dimensions.

Let us take the equations of motion of a perfect gas

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}\qquad (uvw,\ xyz) \tag{1}$$

$$\frac{\partial \ln\rho}{\partial t}+u\frac{\partial \ln\rho}{\partial x}+v\frac{\partial \ln\rho}{\partial y}+w\frac{\partial \ln\rho}{\partial z}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)+u\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right)+v\frac{\partial}{\partial y}\left(\frac{p}{\rho^\gamma}\right)+w\frac{\partial}{\partial z}\left(\frac{p}{\rho^\gamma}\right)=0 \tag{3}$$

The notations are the same as in [2]. Let the set of functions

$$u=u'(t,x,y),\quad v=v'(t,x,y),\quad w=\frac{z}{t-a},\quad \rho=\rho'(t,x,y),\quad p=p'(t,x,y) \tag{4}$$

satisfy the system of equations (1) to (3).

Then the set of functions

$$u = u_0(t, x, y) = \frac{a-\tau}{b} u'(\tau, \xi, \eta) + \frac{\xi}{b}, \quad \rho = \rho_0(t, x, y) = \left(\frac{a-\tau}{b}\right)^3 \rho'(\tau, \xi, \eta) \quad (5)$$

$$v = v_0(t, x, y) = \frac{a-\tau}{b} v'(\tau, \xi, \eta) + \frac{\eta}{b}, \quad p = p_0(t, x, y) = \left(\frac{a-\tau}{b}\right)^5 p'(\tau, \xi, \eta)$$

will satisfy the system of equations of two-dimensional gas motion which is obtained from the system (1) to (3) if the third of Equations (1) is discarded, we set $w = 0$ and consider all functions independent of z. Hence

$$\tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c} x, \qquad \eta = \frac{b}{t-c} y \qquad (6)$$

and the adiabatic index is $^5/_3$. The a, b, c, in all the formulas written above are arbitrary constants with the dimensionality of time. Their validity can be shown by direct substitution of (5) and (6) into the mentioned equations of two-dimensional gas motion, and taking into account here the form of (1) to (3) used when substituting the solution (4).

It can be shown analogously that the solution of the system (1) to (3)

$$u = u'(t, x), \quad v = \frac{y}{t-a}, \quad w = \frac{z}{t-a}, \quad \rho = \rho'(t, x), \quad p = p'(t, x)$$

corresponds to the solution of the one-dimensional gas motion equations

$$u = u_0(t, x) = \frac{a-\tau}{b} u'(\tau, \xi) + \frac{\xi}{b}, \qquad \rho = \rho_0(t, x) = \left(\frac{a-\tau}{b}\right)^3 p'(\tau, \xi)$$

$$p = p_0(t, x) = \left(\frac{a-\tau}{b}\right)^5 p'(\tau, \xi), \qquad \tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c} x$$

with adiabatic index again equal to $^5/_3$.

In the same manner, it may be found that if the set of functions

$$u = u'(t, x), \qquad v = \frac{y}{t-a}, \qquad \rho = \rho'(t, x), \qquad p = p'(t, x)$$

satisfies the system of two-dimensional gas motion equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (uv, xy)$$

$$\frac{\partial \ln \rho}{\partial t} + u\frac{\partial \ln \rho}{\partial x} + v\frac{\partial \ln \rho}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right) + v\frac{\partial}{\partial y}\left(\frac{p}{\rho^\gamma}\right) = 0$$

then the set of functions

$$u = u_0(t, x) = \frac{a-\tau}{b} u'(\tau, \xi) + \frac{\xi}{b}, \qquad \tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c} x \qquad (7)$$

$$\rho = \rho_0(t, x) = \left(\frac{a-\tau}{b}\right)^2 \rho'(\tau, \xi), \qquad p = p_0(t, x) = \left(\frac{a-\tau}{b}\right)^4 p'(\tau, \xi) \qquad (8)$$

will satisfy the system of equations corresponding to one-dimensional gas motion with the adiabatic index 2 .

For example, let us have the solution of the two-dimensional problem in the form

$$u' = \frac{x+\mu}{t+\nu}, \quad v' = \frac{y}{t-a}, \quad \rho' = \rho'(t), \quad p' = p'(t)$$

Here μ, ν are constants. According to (7) and (8), the solution of the one-dimensional gas motion equations

$$u = \frac{a-\tau}{b}\frac{\xi+\dot{\mu}}{\tau+\nu} + \frac{\xi}{b} = \frac{b^{\mu}+(a+\nu)x}{(a+\nu)(t-c)-b^2} = \frac{x+\mu'}{t+\nu'}$$

$$\rho = \left(\frac{b}{t-c}\right)^2 \rho'(t), \qquad p = \left(\frac{b}{t-c}\right)^4 p'(t) \qquad \left(\mu' = \frac{b\mu}{a+\nu}, \quad \nu' = -\left(c + \frac{b^2}{a+\nu}\right)\right)$$

corresponds to it.

Hence, a one-dimensional gas motion has been obtained with the same law of velocity u change, but with some other time dependence of the pressure and density. The solution obtained may be considered as a solution of the one-dimensional Cauchy problem for which the gas state (for $\nu' > 0$)

$$u = \frac{x+\mu'}{\nu'}, \qquad \rho = \rho_0 = \left(\frac{b}{c}\right)^2 \rho'(0), \qquad p = p_0 = \left(\frac{b}{c}\right)^4 p'(0)$$

is given at the initial instant $t = 0$.

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